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ABSTRACT

This paper examines sixth-grade children's thinking related to solving ratio and proportion problems before these children had received any instruction in the domain. It explores unitizing and norming, one of the processes identified by both content analyses (Behr, Harel, Post, & Lesh, 1990); Harel, Behr, Post & Lesh, 1990) and research in children's thinking (Steffe, 1990; Lamon, 1989) as a possible approach for improving a mechanism for the growth of mathematical thinking. Children's use of this approach for ratio and proportion problems was examined and the results analyzed as a counterpoint to development of units in both the additive and multiplicative conceptual fields. The study indicated that children came into instruction in ratio and proportion with some informal understanding. While none of the children used symbols to represent their thinking, their conceptual competency exceeded their symbolic competency. Successful student strategies consistently involved a strong intuitive application of unitizing and norming. (KR)

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# Ratio and Proportion: Cognitive Foundations in Unitizing and Norming

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# Ratio and Proportion:
# Cognitive Foundations in Unitizing and Norming

A major contribution of cognitive science to mathematics education has been knowledge of the cognitive processes students use in mathematical problem solving. In several well-defined mathematical domains (Carpenter & Moser, 1983; Hiebert & Wearne, 1985; Post, Wachsmuth, Lesh, & Behr, 1985), the rich descriptions of children's thinking processes in relation to specific content bear considerable potential for enhancing the classroom teacher's pedagogical content knowledge and ability to assess and teach for meaningful learning. Knowledge of the interplay between content and thinking frameworks may, in fact, influence the teacher's entire instructional decision-making process (Carpenter & Fennema, 1988) thus providing an interface between research on children's thinking and research on teaching (Fennema, Carpenter & Lamon, 1988).

The current psychological Zeitgeist provides further impetus to increase the number of domains in which detailed analyses of children's thinking are available. Constructivism, the belief that people construct knowledge for themselves, has cast an aura of illegitimacy over the prescription of content strictly according to mathematical structure and has advanced the viewpoint that individual students' existing interpretation and organization of mathematical experiences determines the appropriateness of further instruction.

Research Goals in Complex Domains

There is a question as to whether such detailed characterizations of children's thinking are possible within complex mathematical domains. The domain of ratio and proportion, for example, is one complex domain where such a body of knowledge is desirable. The domain represents a critical juncture at which many types of mathematical knowledge are called into play, and a point beyond which a student's understanding in the mathematical sciences will be greatly hampered if the conceptual coordination of all the contributing domains is not attained. Proportional reasoning plays such a critical role in a student's mathematical development that it has been called a watershed concept, a cornerstone of higher mathematics and the capstone of elementary concepts (Lesh, Post & Behr, 1988).

In most complex mathematical domains, however, it is impossible to designate a single, linearly ordered path through the many content domains that are essential to its understanding. Development looks more like a tree with an intricate branching system. Prior knowledge and experience are so extensive and interactive that how students have arrived at any given point in their learning is not the important question to ask. The compelling questions related to complex domains are: what do children know before instruction; what can instruction do to facilitate learning; and how do we recognize it when it occurs? Thus, rather than trying to map out optimum routes through the content of elementary school mathematics à la Gagne, a more

rewarding enterprise, if successful, will be the identification of mathematical processes that facilitate growth in mathematical thinking both in horizontal and vertical directions.

An important consideration is connectivity. The logical status of many topics within rational numbers, for example, is of such complexity that it necessitates not only an understanding of many prerequisite domains, but also an ability to relate them to each other (Streefland, 1983). Thus, one of the goals of current research is to identify important mathematical processes, themes or connections by which thinking becomes progressively more sophisticated from early childhood through early adulthood. We might call these "mechanisms" by which more advanced reasoning evolves. This pursuit involves the investigation of complex domains with the goal of discovering continuities or discontinuities with earlier learned mathematical structures and/or processes. Under this approach, for example, one might ask, "How can we view ratio, and eventually proportional reasoning, as an extension of some basic mathematical idea(s)?" "What arithmetic knowledge may be useful in developing the more complex domains encountered in middle school?" "What intuitive, informal, or existing knowledge aids the learning of rational number concepts?" That is, researchers are seeking basic conceptual schema that comprise part of the backbone of mathematics, processes that are recurrent, recursive and/or of increasing complexity across mathematical domains.

In a search for some viable way to construe the

multiplicative conceptual field, two valuable lines of research have been seeking such mechanisms for mathematical growth. Research within complex mathematical domains is being conducted using both top-down and bottom-up approaches. Theoretical mathematical and semantic analyses of rational number topics (Behr, Harel, Post, & Lesh, 1990; Harel, Behr, Post, & Lesh, 1990) are based on the assumption that the failure of the current curriculum provides a limited perspective of the multiplicative conceptual field, and that understanding will require a broader range of experiences. This entails a thorough analysis of the quantities (both units of measure and magnitudes) germane to multiplicative situations. A second methodology examines children's knowledge in clinical interviews either before formal instruction (Lamon, 1989) or after limited instruction (Mack, 1990) or in teaching experiments (Steffe, 1990) to determine which informal (intuitive, child-generated, previous or existing) knowledge forms a useful foundation upon which instruction might be built.

This paper examines sixth grade children's thinking in the process of solving ratio and proportion problems before these children had received any instruction in the domain. Specifically, it explores unitizing and norming, one of the processes to which both content analyses (Behr, Harel, Post, & Lesh, 1990; Harel, Behr, Post, & Lesh, 1990) and research in children's thinking (Steffe, 1990; Lamon, 1989) are pointing as a mechanism for the growth of mathematical thinking. After a

discussion of unitizing and norming, children's use of this mechanism in relation to ratio and proportion problems will be examined. These results will then be analyzed as a counterpoint to development of units in both the additive and multiplicative conceptual fields.

Unitizing and Norming

The ability to construct a reference unit or a unit whole, and then to reinterpret a situation in terms of that unit, appears critical to the development of increasingly sophisticated mathematical ideas. This process, begun in early childhood (Steffe & Cobb, 1988), involves the progressive composition of units to form increasingly complex quantity structures.

The notion that higher order grouping allows a student to view aggregates as well as individual members of a set, and thus to gain mathematical power, has been demonstrated in many areas, including the acquisition of early counting strategies (Steffe & Cobb, 1988), children's partitioning strategies (Pothier, 1981), and the acquisition of early addition and subtraction strategies (Carpenter and Moser, 1983).

The process of forming composite units probably begins visually in early childhood quantifying activities such as subitizing, and is then extended into counting activities. When a child is counting on his/her fingers and begins to substitute the counting word "five" for the fingers on one hand, the child has adopted a more powerful grouping technique; likewise, when

the child counts on from the first number in an addition problem, or counts by tens, we can interpret his/her strategy as an indication of a higher level conceptual organization.

Research in the natural development of language hierarchies (Callanan & Markman, 1982; Markman, 1979) has substantiated this perspective, suggesting that more sophisticated thinking results when one reframes the situation in terms of a more collective unit because it invokes a part/whole schema which allows the student to think about both the aggregate and the individual items that compose it.

Advancing from addition and subtraction into multiplicative structures, requires the conceptual coordination of multiple compositions. Some of the simplest multiplicative structures require a three-tiered composition of units (Steffe, personal communication, May 17, 1988). One such structure is illustrated in the following example: find 3/4 of 16 objects.

1. Consider the 16 objects as 16 units. You then have 16 one-units.
2. Create units of units, that is, 4 composite units, each consisting of 4 one-units. You then have 4 four-units.
3. Create units of units of units, that is, create one three-unit consisting of three of the four four-units.

In addition to the change in the magnitudes of quantities, most multiplicative situations are referent-transforming compositions (Schwartz, 1988). That is, multiplicative structures combine two magnitudes with different labels to

produce a quantity whose label is not the same as either the multiplicand or the multiplier. The result is an intensive quantity, a new unit of measurement, a special relationship between two extensive quantities. This new quantity needs to be conceptualized as an entity in itself, different from its constituent measures. For example, 5 bags of candy with 6 candies per bag yields 30 candies (not bags of candy and not candies per bag). Thus, multiplicative structures involve several layers of cognitive complexity. While it is not clear that there is a continuous sequence of development from the counting and iteration processes of addition and subtraction into the more complicated intensive measurements units used in multiplicative situations (e.g., miles per hour, candies per bag, etc.), theoretically, at least, it is impossible to avoid some discussion of composite units in the development of these complex quantity structures.

To take Steffe's ideas a little farther, we might think about creating a composite unit and then reinterpreting a given situation in terms of that unit. Freudenthal (1983) used the term "norming" to describe this process of reconceptualizing a system in relation to some fixed unit or standard. For example, imagine that the earth is the size of a pin's head (about 1 mm diameter) and reconceptualize the solar system in terms of that definition. Then the sun would appear as a sphere with a diameter of 10 cm. at a distance of 10 m. from the earth (Freudenthal, 1983). This norming process, or the adoption of

some framework in which to conceptualize a situation, is really quite prevalent in mathematical thinking. Its role in the world of ratio and proportion is examined next.

The Role of Unitizing and Norming in Ratio and Proportion

Both ratios and fractions (a special case of ratios in which parts of the same set are quantified), are the result of multiple compositions. These complex units are then used to reconceptualize other quantities. For example, consider the ratio of 9 people to 2 cars. I will adopt a new notation, namely, 9 ⊖ 2. [Avoiding the standard fraction bar notation, 9/2, may be critical for helping children distinguish fractions and ratios, and the dot notation, 9:2, while it may present a clear enough distinction for American children, presents problems for some European children. The notation : sometimes means "divided by;" hence, 9:2 fails to be distinguished from the quotient interpretation of fraction (deLange, Personal Communication, April, 1990).]

9 ⊖ 2

9 is a composite unit: 9 individuals are considered as one 9-unit, a unit of units. Likewise, 2 is a composite unit: 2 individual cars become a single 2-unit.

A ratio within a single measure space, for example, the

comparison of 5 feet to 17 feet, can be conceptualized as growth along a single dimension. In this case, one or the other of the components of the ratio itself can be considered the unit whole in terms of which the other quantity can be interpreted.

Interesting consequences also arise when a ratio itself is considered the unit whole, that is, when we view the ratio as a complex index in terms of which other ratios may be created. In the following example, members of an equivalence class of a particular ratio are of interest, the ratio 9 ⊖ 2 is the norm for reinterpretation, and equivalent ratios are the result of forming units of units:

18 ⊖ 4 is 2 of the unit 9 ⊖ 2.
Symbolically, 18 ⊖ 4 = 2 ( 9 ⊖ 2 ).

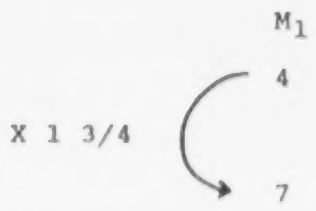
In rational numbers, we not only create new unit wholes by composing units again and again, but we also norm against those

unit wholes. For example, consider the processes of dividing fractions and comparing fractions with understanding. (Fractions are merely a special kind of ratio in which the numbers being compared both quantify different portions of the same set of objects.) Fraction division involves the reconceptualization of one fraction in terms of the other:

| | |
|---|---|
| 3/4 ÷ 1/2 | If 1, the fractions are equivalent. |
| How many 1/2's in 3/4 ? | If > 1, 3/4 > 1/2. |
| | If < 1, 3/4 < 1/2. |

| | |
|---|---|
| unit whole | 3/4 = 1(1/2) + 1/2 (1/2) |
| Answer: 3/4 ÷ 1/2 = 1 1/2 | 3/4 > 1/2 |

The determination of a scale factor within a measure space entails a reinterpretation of one measure in terms of the other using the process of scalar decomposition:

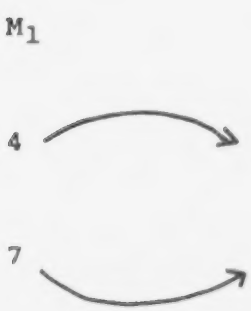



7 = 1 (4) + 3/4 (4)

In this case, the four was chosen as the unit whole and 7 was

written in terms of that 4-unit.

In a proportion, solution for a missing term by scalar decomposition of a function involves a norming process:




$8\ 3/4 = 1(7) + 1/4\ (7)$

$f(x) = 1(x) + 1/4\ (x)$

$1(4) + 1/4\ (4) = 5$

The process of norming can achieve yet another level of sophistication when, for reasons of uniformity and/or convenience, an independent unit is chosen as the standard for norming. This is precisely what we do in calculating percentages.

In short, it appears that the analysis of ratio and proportion through a framework of unitizing and norming may provide some useful insights into the critical relationships we would like children to understand. The ability to reinterpret information in terms of different unit wholes--sometimes several times within the same situation--appears to be essential to understanding ratios. More specifically, the ability to advance beyond norming with a single unit whole (composite or otherwise) to reconceptualizing a situation in term of a new kind of unit, which itself consists of a complex relationship between two quantities, appears to be critical to proportional reasoning.

The question remains as to how closely this rational analysis conforms to what children actually do as they encounter the domain. After a brief review of existing research, children's preinstructional thinking is analyzed next.

## Research in Ratio and Proportion

Though research with preadolescent students is not extensive, it appears that student representations of situations involving ratio and proportion occur on an informal, qualitative basis long before those students are capable of treating the topic quantitatively (Tourniaire, 1986; Treffers and Goffrey, 1985; Van den Brink & Streefland 1979; Steffe, 1969; Steffe & Parr, 1968), and that many student strategies and processes develop independent of instruction (Post, Behr, Lesh, & Wachsmuth, 1985). That is, children naturally employ a form of mathematical intuition, an informal knowledge system (Kieren, 1983; Treffers & Goffree, 1985; Streefland, 1984, 1985). This informal knowledge includes a visual understanding of ratio and proportion (Steffe & Parr, 1968), especially of congruence and similarity (Van den Brink & Streefland, 1979). In addition, children use self-invented solution strategies, which are primitive, context-bound, relatively symbol-free, and based upon counting, adding and halving (Hart, 1984). Some fairly sophisticated student-invented methods have been documented (e.g., see Rupley, 1981.)

Although previous studies indicate that children's thinking

in relation to ratio and proportion problems can be reasonable, productive, and even insightful, they have provided little information as to how, and when, and in response to which situations, a child's predominantly additive view of the world changes to include a multiplicative perspective. A very basic question remains unanswered: At what point can we say a child reasons proportionally? What kinds of relationships need to be grasped in order to claim that a someone understands ratio and proportion? Could an interpretation of children's thinking in terms of a unitizing/norming framework provide any new insights?

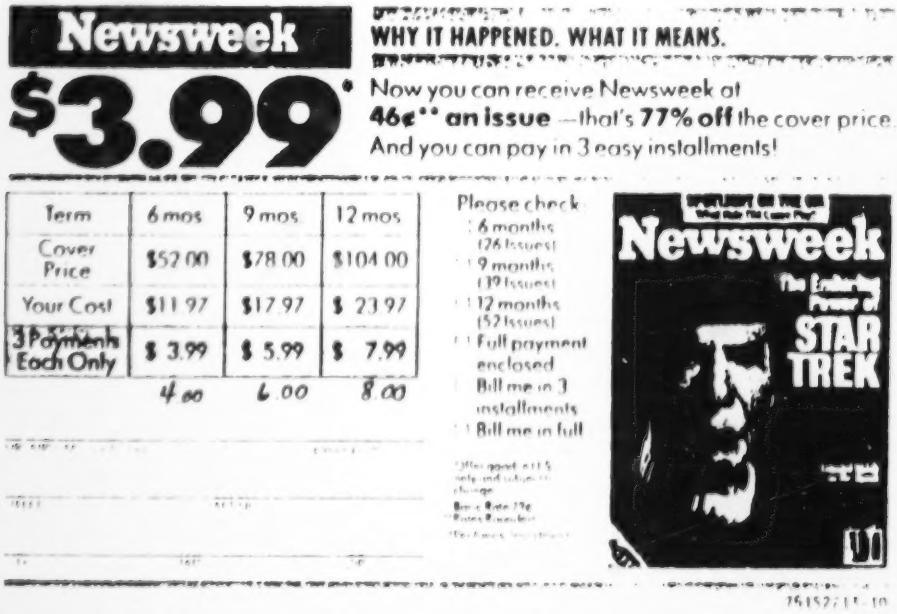
In a recent study, Lamon (1989) engaged twenty-four sixth-grade children, who had not yet received formal instruction in ratio and proportion, in audio-taped clinical interviews. In a priori theoretical analyses of ratio and proportion situations, she identified four semantic problem types and five mathematical notions essential to proportional reasoning. The resulting content X problem type framework was used to put parameters on the ill-defined domain, to guide the construction of problem situations for the clinical interviews, and to provide explanatory power for the phenomena observed in the student interviews. The ensuing discussion focuses on a special subset of the problem situations those children addressed, in particular, five problems which were designed to elicit information regarding the children's power of unitizing and norming.

Children's Thinking: Unitizing and Norming

Children's encounters with the following problems will be analyzed: 1) the balloon problem; 2) the subscription problem; 3) the apartment problem; 4) the pizza problem; and 5) the alien problem.

The balloon problem. Ellen, Jim and Steve bought 3 helium filled balloons and paid $2.00 for all three. They decided to go back to the store and get enough balloons for everyone in their class. How much did they have to pay for 24 balloons?

The subscription problem. The following subscription card was taken from a magazine.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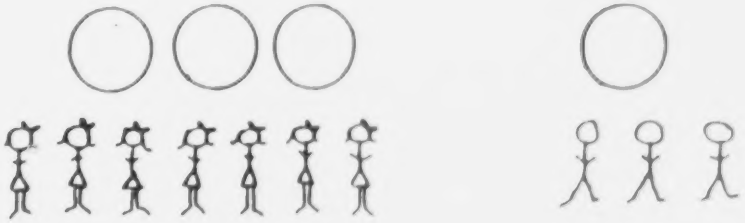
The apartment problem. In a certain town, the demand for rental units was analyzed and it was determined that to meet the communities needs, builders would be required to build units in the following way: every time they build 3 single units, they should build 4 two-bedroom units and 1 three-bedroom unit. Suppose a builder is planning to build a large apartment complex containing between 30 and 40 units. How many units should he/she build to meet this regulation?

Suppose he/she built 32/40 (choose one). How many one-bedroom units, two-bedroom units and three-bedroom units would the apartment building contain?

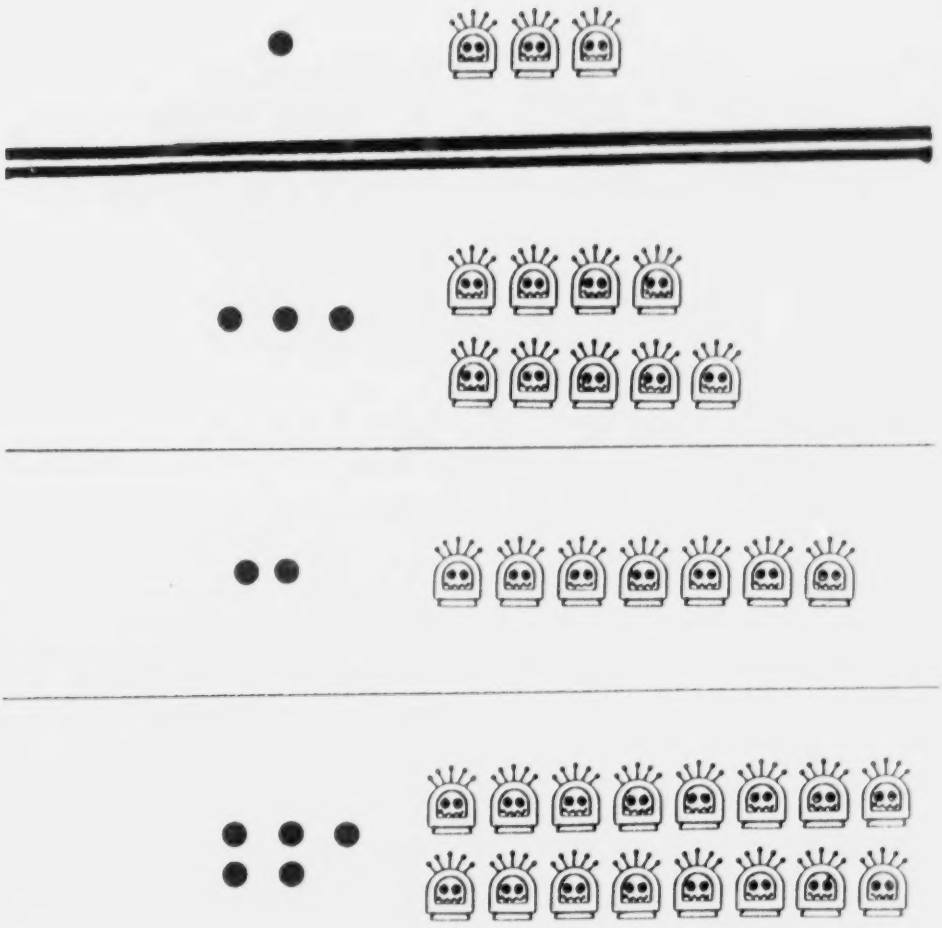
Suppose he/she had built 14 one-bedroom apartments, 18 two-bedroom apartments and 4 three-bedroom apartments. Would the requirement be satisfied?

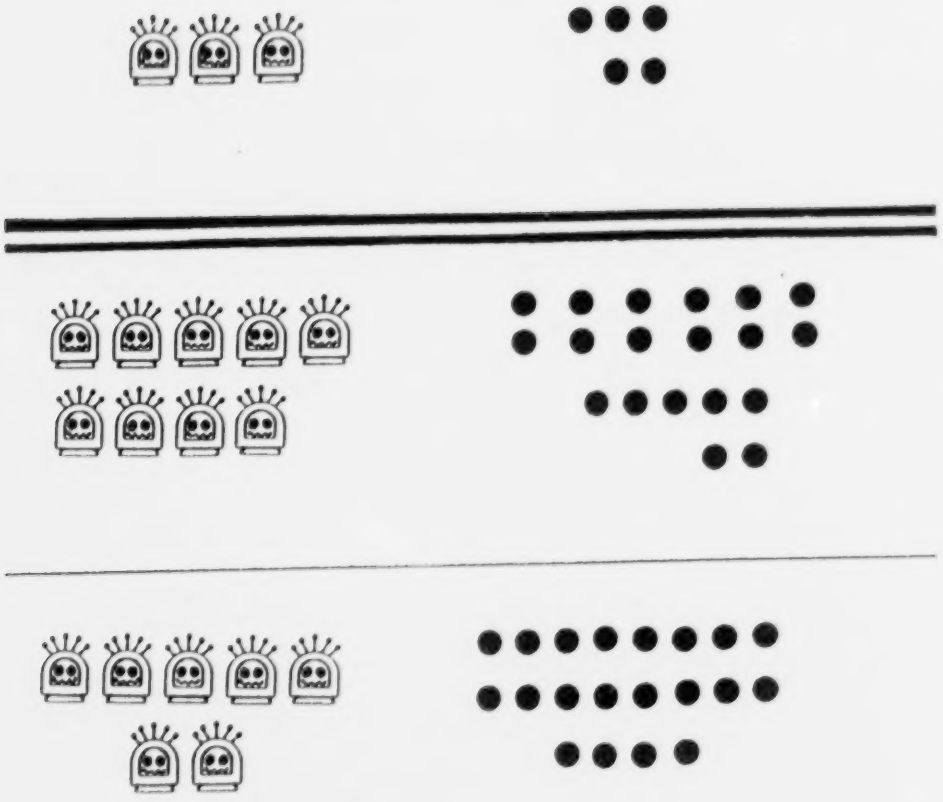
The pizza problem. Consider the following drawing of children and pizzas.



Who gets more pizza, the girls or the boys?

The alien problem. Consider the following pictures of aliens and food pellets.





At the top of each page, you can see some aliens and the number of food pellets they need to live for one day. Assume that all of the aliens eat about the same amount. Tell me if each group of aliens under the double line has the right number of food pellets, too many, or too few.

(Referring to the second picture:) Suppose I had 36 food pellets. How many aliens could I feed for one day?

## Results

The study indicated that children come into instruction in ratio and proportion with some informal understanding. While none of the children used symbols to represent their thinking, their conceptual competency exceeded their symbolic competency. Successful student strategies consistently involved a strong intuitive application of unitizing and norming. Consider, for example, the successful processes used to solve the balloons problem. The number of students who used each of these processes is given in parentheses.

> Ellen, Jim and Steve bought 3 helium filled balloons and paid $2.00 for all three. They decided to go back and get enough balloons for everyone in their class. How much did they have to pay for 24 balloons?

(10) 1. For every third balloon, you pay $2.00. So you have 8 packets (sets, groups, or bunches) and 8 X 2 would be $16.

(2) 2.

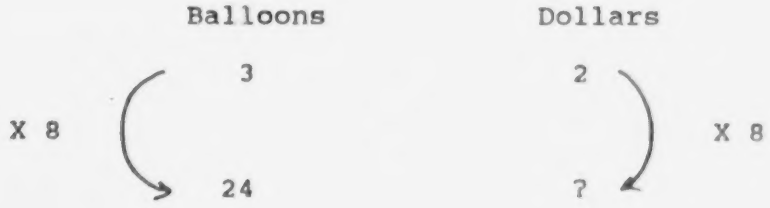
| | |
|---|---|
| 3 | $2.00 |
| 6 | 4.00 |
| 9 | 6.00 |
| 12 | 8.00 |
| 15 | 10.00 |
| 18 | 12.00 |
| 21 | 14.00 |
| 24 | 16.00 |

(3) 3. $2.00/3 = .6666... or 66 r.2 .66 X 24 = $ 15.84

(1) 4. 3 balloons were $ 2.00 and 2 divided by 3 is 2/3, so I asked myself how many 24ths is 2/3? The answer is 16.

In subsequent questioning, the two students who used the building up strategy (shown above in # 2) failed to show any

understanding of the functional relationship in this situation. The students who used the two-step unit rate strategy (shown above in # 3) sacrificed accuracy; although correct, their thinking in terms of single units proved less powerful than thinking in terms of composite units. Without using the traditional symbolism, the last student engaged in proportional reasoning with full recognition of the structural relationships by equating rates. The first response indicates, however, that for the majority of the students who solved the problem, it was their ability to conceptualize the situation in terms of groups, or sets, or packets, or bunches--various class terms were used--that enabled them to solve the problem. Essentially, they solved a proportion consisting of within measure space ratios by applying the scalar operator between members of the first measure space to the second measure space:




The subscription situation provided an interesting forum in which to investigate students' thinking because it provided a multitude of divisibility relationships, both within and between measure spaces, around which the student could base his/her solution.

> The student was shown a card in a magazine which offers three plans for subscribing to the magazine: 1) you may subscribe for a 6-month period, during which time you will receive three bills each for the amount of $4.00, 2) you may subscribe for 9 months and receive 3 bills of $6.00 each, or 3) you may subscribe for 12 months and pay 3 bills of $8.00 each.
>
> Do you get a cheaper rate if you buy the magazine for a longer period of time?

When the explicit rates as well as the comparable one-month, and three-month rates are considered, this situation might be summarized as follows:

| $M_1$ | $M_2$ | |
|---|---|---|
| subscription time in months | cost in dollars each payment | total |
| 1 | 2/3 | 2 |
| 3 | 2 | 6 |
| 6 | 4 | 12 |
| 9 | 6 | 18 |
| 12 | 8 | 24 |

The 13 students who were successful with this problem used interesting solution processes; their solutions follow. Again, the number of students who used each solution is given in parentheses.

(5) 1. The price for two 6-months is the cost of 12 months, and 9 is just the middle of that.

(1) 2. 4 is 2/3 of $6.00 and 6 is 2/3 of 9.

(1) 3. Three payments of $4.00 is $12.00; three payments of $6.00 is $18.00; three payments of $8.00 is $24.00. Divide each in half and you get 6, 9, and 12.

(1) 4. The months go up by 3 and the dollars go up by 6, so it's the same any way you do it.

(1) 5. In each case, if you figure out how much it costs a month, you get $2.00 a month.

(4) 6. For the 6 months, every 3 months would be $2.00; for the 9 months, every 3 months is $2.00; the same for 12 months.

Highlighting the cost of one payment tended to distract the student's attention from the fact that the total cost in dollars was twice the number of months of subscription. Thus, only one student noticed the functional relationship. In the fourth solution, the student confirmed that the rate must be the same merely by identifying patterns. The second solution identified and compared the scalar operators in both measure spaces. However, the majority of the solutions involved finding and comparing unit rates. It is interesting to note that of the 11 solutions using unit rates, only one student used a 1-month unit. The other solutions used a 3-month or a 6-month unit. It appears that the students made decisions about the most appropriate unit with which to reinterpret a situation, based on the specific conditions of the problem. In this case, most students chose the 3-month unit as the most efficient unit because 3 was the greatest common factor of 6, 9, and 12.

Students' thinking about the apartment problem proved quite interesting when analyzed in terms of the formation of composite units. This problem requires the repeated formation of composite units, and then the decomposition of those units into its

constituent parts again.

> In a certain town, the demand for rental units was analyzed and it was determined that to meet the community's needs, builders would be required to build units in the following way: every time they build 3 single units, they should build 4 two-bedroom units and 1 three-bedroom unit. Suppose a builder is planning to build a large apartment complex containing between 30 and 40 units. How many units should he/she build to meet this regulation?
>
> Suppose he/she built 32/40 (choose one). How many one-bedroom units, two-bedroom units and three-bedroom units would the apartment building contain?

This problem may be analyzed as follows:

(units) 1-bedroom apartments
2-bedroom apartments
3-bedroom apartments

(units of units) 3 1-bedroom apartments
4 2-bedroom apartments
1 3-bedroom apartments

(units of units of units) Those apartments form one 8-unit.

(units of units of units of units) How many 8's are needed?

4 or 5 of that 8 unit

(decomposition of composite units) $40 = 5\ (8) = 5\ (1 \Theta 3 \Theta 4) =$
$5 + 15 + 20$

Fifteen out of 24 students were able to think through this situation. Eleven students used the composition and decomposition process described above. Laurie's protocol is typical of the solution strategies they used.

Laurie: He should build 15 one-bedroom apartments, 20 two-bedroom apartments and 5 three-bedroom apartments.

I: Tell me how you thought about that.

Laurie: Well, 3 and 4 and 1 make like one building and you can make 5 buildings. So there would be 5 of each kind of apartment.

I: What do you mean 5 of each kind?

Laurie: There would be 3 one-bedrooms in each building, and 5 buildings, so that's 15 one-bedrooms and like that for the other sizes.

Four students did not create one 8-unit, but instead, simultaneously added columns of 3's, 4's and 1's or the multiples of 3, 4, and 1 until they achieved a total of 32 or 40 apartments, and then noticed that they had 4 or 5 of each type. Kari's protocol illustrates this solution process.

Kari: (Worked for a few minutes on paper.) 15 one-bedroom apartments, 20 two-bedroom apartments and 5 three-bedroom apartments.

I: Would you explain that answer? What were you doing here?

Kari: (Showed the following work.)

| | | | |
|---|---|---|---|
| 3-1 bed | 6 | 9 | 15 |
| 4-2 bed | 8 | 12 | 20 |
| 1-3 bed | 2 | 3 | 5 |
| | 16 | 24 | 40 |

First I doubled each number and added them up. That was 16, so I could get more...So I did three more of each and I just added them up. Then I decided to try each one times 5 and that gave me 40.

In addition to the fifteen successful strategies, there were two more which completed the composition of units (that is, decided that there were going to be 32 or 40 total apartments), but were unable to reverse the process and decompose into the numbers of one, two and three-bedroom apartments.

An additional question further probed some students' understanding of the situation:

> Suppose he built 14 one-bedroom apartments, 18 two-bedroom apartments and 4 three-bedroom apartments. Would the regulation be met?

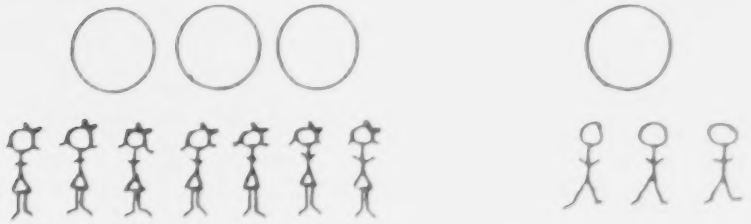
Predictably, the students who were able to decompose the units in the first question were able to decide that, in fact, the regulation would not be met. The two students who were not able to decompose were also not able to recognize that the total number of apartments was not a multiple of 8, nor were the numbers of apartment types divisible by the same factor.

Sometimes, students' reasoning defied expected patterns. Divisibility relationships, integral ratios and unit rates, for example, are thought to facilitate symbolic solutions of proportions (Karplus, Pulos, & Stage, 1983; Noelting, 1980a, 1980b), but for these students who were not yet operating symbolically, the numerical relationships did not appear to be of major importance. Instead, these students used both ratios/rates as units and operated one upon the other, using a double matching process.

Student strategies for the pizza problem, for example, show

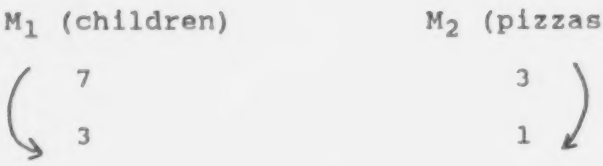
that their thinking did not involve either of the methods for setting up ratios predicted by dimensional analysis: between or within measure spaces. Instead, students maintained the separateness of the sets even after they had formed ratios and had begun to operate with them. Consider the pizza problem and some solutions to it.

> Student is shown the following drawing of children and pizzas.
>
> Who gets more pizza, the girls or the boys?



Dimensional analysis would predict two ways in which to think about this situation. There are two measure spaces, one measuring the cardinality of the set of children, and the other measuring the cardinality of the set of pizzas, and one way to relate them is with a between strategy:

The following rates would then be compared: 7/3 and 3/1. A second option is to think about children and pizzas separately, in which case quantities would be related using a within strategy.




Consequently, the following ratios would be compared: 7/3 and 3/1. (Merely by coincidence, rates and ratios turn out to be the same for this particular problem.)

To compare, the expected method would be to first notice that the ratio/rate 3/1 is a unit rate, and then to determine what should be the corresponding ratio/rate for equivalence (Karplus, Pulos, & Stage, 1983). This process involves norming in terms of the unit rate which, in this case, would produce 9/3 as the desired equivalent. At this point, the student could determine whether the ratios/rates are equivalent, but needs to invoke additional knowledge to determine the direction of the inequality. He/she might switch to a division interpretation of the fraction symbol, or apply other informal knowledge concerning the size of fractions with like denominators to determine that $7/3 < 9/3$.

Eighteen of the 24 students interviewed in this study solved the problem, but not in the expected manner. They all used the same solution strategy explained by Kari.

I: Who gets more pizza, the girls or the boys?

Kari: The girls.

I: Why do you think the girls do, Kari?

Kari: Because the boys have to share 3 people with 1 pizza and if the girls did the same thing...if they had 3 people to this pizza (pointed to the first pizza and covered 3 girls) and three people to this pizza (pointed to the second pizza and covered three more girls), then 1 person would have a whole pizza to themselves. So some people could go over to that one to get some more.

This process might be analyzed into the following steps:

1. Distinguish four kinds of units.

| | |
|---|---|
| girls | boys |
| girl pizzas | boy pizzas |

2. Create units of units.

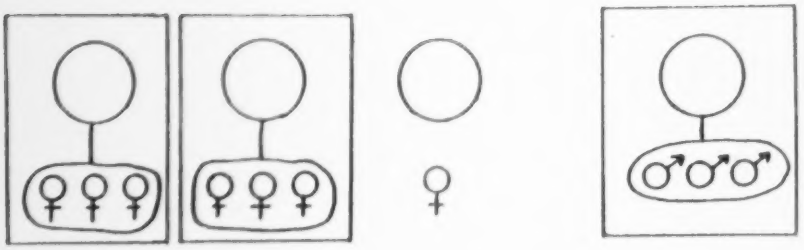
| | |
|---|---|
| 7 girls | 3 boys |
| 3 girl pizzas | 1 boy pizza |

3. Create units of units of units. Here, thinking involves a between strategy and the rate becomes the new unit.

| | |
|---|---|
| 7 ⊖ 3 | 3 ⊖ 1 |

4. Choose one rate as the standard unit and reinterpret the other in terms of that unit. In this case, the rate 3:1 became the standard unit.

How many 3 ⊖ 1 units are in 7 ⊖ 3?



5. Since 1 1 yields more pizza per person than 3 ⊖ 1, the girls get more pizza.

Symbolically, this process might be represented in the following way:

$$7 \ominus 3 = 2\ (3 \ominus 1) + 1 \ominus 1 = 6 \ominus 2 + 1 \ominus 1$$

$$1 \ominus 1 > 3 \ominus 1$$

The fact that 3 ⊖ 1 is a unit rate may or may not have been critical in its selection as the norming unit. However, students used that rate in a far different manner than would have been predicted. Both of its components were compared simultaneously with the components of the other rate.

All 24 students solved the aliens problem in the same way, that is, using the double matching strategy. The following notation might be used to describe the process they used:

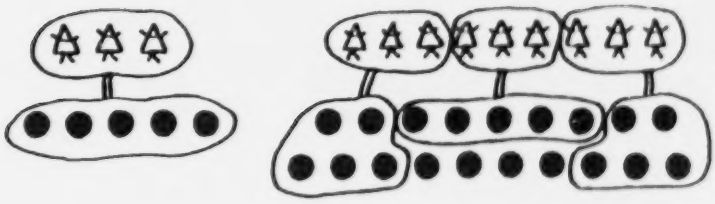
| | |
|---|---|
| group 1 of aliens | group 2 of aliens |
| group 1 food pellets | group 2 food pellets |

$3A_1$ $9A_2$

$5P_1$ $19P_2$

$3 \theta 5$ $9 \theta 19$

How many $(3 \theta 5)$'s are in $(9 \theta 19)$?



Answer: $9 \theta 19 = 3(3 \theta 5) + 0 \theta 4 = 9 \theta 15 + 0 \theta 4$ ← These are extras.

Again, the divisibility relationship between $3A_1$ and $9A_2$ probably led to the use of $3 \theta 5$ as the norming unit.

The purpose of the follow-up question to the alien problem was to see if a student who solved by norming with the ratio/rate as a unit could reverse their thinking. Not all students were asked the questions, but of those who were, two students solved the aliens problem using a building up process as described below.

> Suppose I had 36 food pellets. How many Martians could I feed for one day? (Prior information told the student that 3 aliens could live on 5 food pellets for one day.)

Chris: Well, 3 aliens need 5 pieces of food, so about 3 times it would be...9 could use 15 (wrote 9 and 15.)...and timesed again, another 9 would get 15 (wrote 9 under the first 9, 15 under the first 15). So, we're up to 30. That means we could use 5 more for 3 more aliens...and there would be 1 left over (wrote 5 and 3 in the appropriate columns and added both columns). So we have enough food for 21 aliens.

Symbolically, Chris' thinking could be represented in the following way:

$3(3 \Theta 5) + 3(3 \Theta 5) + 1(3 \Theta 5) + (0 \Theta 1) = 9 \Theta 15 + 9 \Theta 15 + 3 \Theta 5 + 0 \Theta 1 = 21 \Theta 36$

Discussion

An analysis of students' processes for solving these problems revealed that three fourths of the students interviewed naturally formed ratios of otherwise unrelated sets and engaged in the process of norming, or reinterpreting one ratio in terms of the other. The presentation of the situation in the form of pictures probably facilitated the double matching process. Naturally, some presentations make the semantic structure of a problem more transparent than others, and are more likely to optimize student performance. Also, Case's (1978, 1980) argument that easing the load on the working memory facilitates reasoning may be applicable in this case. For whatever reason, students' ability to think about a ratio as an invariant composite unit and to work simultaneously with both components in a double matching process (covariance), illustrated the kind of understanding we would like

them to have about the meaning of ratio and proportion. Their thinking may be interpreted as a kind of pre-symbolic quantitative proportional reasoning. Through the use of presentations and representations that make the norming process transparent, it is likely that students' understand of the notions of ratio and proportion and their connections would be enhanced. Modeling of this nature may be a key to understanding important relationships in the domain.

Although the students did not use symbolic representations, their thinking showed that were quantifying relationships. Even more significantly, they were performing arithmetic operations on those quantities. The extension of this quantification and arithmetic into symbols, appeared imminent. In particular, student thinking was easily translatable into the representation $a \Theta b$ and its appropriate arithmetic operations:

$$(a \Theta b) \pm (c \Theta d) = (a \pm b, c \pm d)$$

$$c\,(a \Theta b) = (ca \Theta cb)$$

$$(a \Theta b) \,/\, (c \Theta d) = M(c \Theta d) + (a - Mc \Theta b - Md)$$

where $M = \max ( [a/c], [b/d] )$ and $[x/y]$ represents the greatest integer in the quotient $x/y$.

Students representations of the balloon, and subscription problems showed that most students not only think in terms of composite units, but also make decisions about which unit is most appropriate. The most simplistic solution to both problems would have been to use a 1-unit as the norming unit, but in most cases, students saw an advantage to using a composite unit: a 3-unit of

balloons and a 3-unit or a 6-unit period of subscription. Thus, levels of sophistication could also be distinguished in children's solutions to these problems based on whether or not they formed composite units and reinterpreted the problems in terms of those units, as well as which units they chose; some units were more efficient than others. This observation would dictate against teaching students to use the two-step one-unit ratio strategy as a standard approach to proportion problems (Herron & Wheatley, 1978; Goodstein & Boelke, 1980; Gold, 1978, 1980).

While future research should pursue these ideas, it is apparent that students' preinstructional reasoning provides some basis on which to build an understanding of intensive quantities.

Unitizing and Norming as a Mechanism for Mathematical Growth

Both theoretical analyses and empirical results suggest that the unitizing and norming process may be an important mechanism by which more advanced reasoning evolves. Understanding ratio and proportion depends on one's ability to view a relationship as a single quantity and then to operate with it. Taking a look at the larger scheme of things, however, reveals that understanding the relative nature of quantities in a ratio may be just another level of sophistication built upon an already-complex foundation of unitizing processes. In Figure 1, some of the processes used by children to make sense of ratio and proportion are placed in counterpoint to some of the unitizing

processes employed in early childhood and addition and subtraction, as well as those children use when thinking about multiplication and division. Moving from left to right between the columns and moving from top to bottom within columns traces a route of increasing sophistication in children's thinking (and/or increasing complexity in the types of units required). Thus, those cells in the matrix lying farther to the right and nearer the bottom represent more demanding cognitive activity.

It is particularly significant that middle school children's preinstructional, pre-symbolic efforts at sense-making in ratio and proportion recapitulated the methods they invented and applied to early addition and multiplication concepts. Intuitive counting strategies emerged when pictorial presentations facilitated the coordination of all the quantities involved. In addition to reaffirming the importance of counting and modeling as important sense-making activities, the simultaneous matching process students employed suggests a valuable activity for facilitating understanding of a ratio. "Experiencing" a ratio through a concrete activity such as counting and pairing two sets of objects may be an important prerequisite to abstracting the concept of ratio, an important first step in a learning cycle.

Research on children's thinking in early addition and subtraction concepts (Carpenter & Moser, 1983) and in multiplication (Steffe, 1990) reveals that modeling and counting activities give way to composing units. The purposeful grouping of single items to make a single entity marks this milestone. In

counting several such entities, what is significant is that a single counting word is attached to composite unit, even though its constituent unit elements are evident. In ratio, three aliens and 5 food pellets were matched and the child counted "1 day"; another three aliens were matched with another 5 food pellets and the child counted "2 days", etc.

The next level of sophistication in cognitive activity is "abstracting." At this level, a child is able to operate with the composite unit in a manner oblivious to its constituent parts. Steffe (1990) calls such units "iterable units" and claims that a double counting process such as "3 is 1; 6 is 2; 9 is 3, etc.", or the coordination of two counting schemes, such as "3,6,9,12,13,14," is evidence that units are iterable. In ratio, units were clearly <u>not</u> iterable for the children interviewed. Their operations with the ratios of aliens to food pellets, for example, were highly dependent on their counting each group of aliens and food pellets before matching them.

After constructing abstract numerical composites, even greater sophistication in thinking is enabled when those composites can be meaningfully related to others. In addition and subtraction, in the part-part-whole relationship, the whole is a numerical composite whose parts are also composites. In multiplication, the distributive property relates multiples of the same composite units. In ratio, another level of complexity is added as a child needs to cognitively coordinate multiples of different composite units. We might speculate that some of the

children were not able to complete the apartment problem because of the extreme difficulty entailed in cognitively coordinating multiples of several different composite units.

## Conclusion

These parallels suggest that the unitizing and norming process may be viewed as a spiralling cognitive activity in which increasingly complex unit formations are taken through the operations of modeling and counting, composing, abstracting, and relating. Research efforts should further investigate children's thinking in the more complex mathematical domains, and seek to clarify possible connections with cognitive activities in less complex domains. The identification of such mechanisms capable of enhancing mathematical thinking is a compelling enterprise if we hope to facilitate higher order thinking in our classrooms.

# REFERENCES

Behr, M., Harel, G., Post, T., Lesh, R. (1990). On the operator concept of rational numbers: Towards a semantic analysis. Paper presented at the Annual Meeting of the American Educational Research Association, Boston, MA.

Callanan, M. A., & Markman, E. M. (1982). Principles of organization in young children's natural language hierarchies. Child Development, 53, 1093-1101.

Carpenter, T. P., & Fennema, E. (1988). Research and Cognitively Guided Instruction. In E. Fennema, T. P. Carpenter, & S. J. Lamon (Eds.), Integrating Research on Teaching and Learning Mathematics (pp. 1-17). Madison: Wisconsin Center for Education Research.

Carpenter, T. P., & Moser, J. M. (1983). Acquisition of addition and subtraction concepts. In R. Lesh & M. Landau (Eds.), Acquisition of mathematics concepts and processes (pp. 7-44). Orlando. FL: Academic Press.

Case, R. (1978). Implications of developmental psychology for the design of effective instruction. In A. M. Lesgold, J. W. Pelligrino, S. D. Fokkema, & R. Glaser (Eds.), Cognitive psychology and instruction (pp. 441-463). New York: Plenum Press.

Case, R. (1980). Intellectual development and instruction: A neo-Piagetian view. In A. E. Lawson (Ed.), 1980 AETS yearbook: The psychology of teaching for thinking and creativity (pp. 59-102). Columbus, OH: ERIC Clearinghouse for Science, Mathematics, and Environmental Education.

Fennema, E., Carpenter, T. P. & Lamon, S. J. (Eds.). (1988). Integrating Research on Teaching and Learning Mathematics. Madison: Wisconsin Center for Education Research.

Freudenthal, H. (1983). Didactical phenomenology of mathematical structures. Dordrecht, Holland: D. Reidel.

Gold, A. P. (1978). Cumulative learning versus cognitive development: A comparison of two different theoretical bases for planning remedial instruction in arithmetic. Unpublished doctoral dissertation, University of California, Berkeley.

Gold, A. P. (1980). A developmentally based approach to the teaching of proportionality. In R. Karplus (Ed.), Proceedings of the Fourth International Conference of the Psychology of Mathematics Education. Berkeley, CA: Lawrence Hall of Science.

Goodstein, M. P., & Boelke, W. W. (1980). A prechemistry course on proportional calculation. Columbus, Ohio: ERIC/SMEAC.

Harel, G., Behr, M., Post, T., & Lesh, R. (1990). A scheme to represent the multiplicative conceptual field. Unpublished manuscript.

Hart, K. M. (1984). Ratio: Children's strategies and errors. A report of the strategies and errors in secondary mathematics project. London: NFER-Nelson.

Herron, J. D., & Wheatley, G. H. (1978). A unit factor method for solving proportion problems. The mathematics teacher, 71, 18-21.

Hiebert, J., & Wearne, D. (1985). A model of students' decimal computation procedures. Cognition & Instruction, 2, 175-205.

Karplus, R., Pulos, S., & Stage, E. (1983). Proportional reasoning of early adolescents. In R. Lesh & M. Landau (Eds.), Acquisition of mathematics concepts and processes (pp. 45-90). Orlando, FL: Academic Press.

Kieren, T. (1983). Axioms and intuition in mathematical knowledge building. In J. C. Bergeron and N. Herscovics (Eds.), Proceedings of the fifth annual meeting of the North American chapter of the International Group for the Psychology of Mathematics Education (pp. 67-73). Montreal.

Lamon, S. J. (1989). Ratio and proportion: Preinstructional cognitions. Unpublished doctoral dissertation, University of Wisconsin, Madison.

Lesh, R., Post, T. R., & Behr, M. (1988). Proportional Reasoning. In J. Hiebert & M. Behr (Eds.), Number concepts and operations in the middle grades, (pp. 93-118). Reston, VA: National Council of Teachers of Mathematics.

Mack, N. K. (1990). Learning fractions with understanding: Building on informal knowledge. Journal for Research in Mathematics Education, 21(1), 16-32.

Markman, E. M. (1979). Classes and collections: Conceptual organization and numerical abilities. Cognitive Psychology, 11, 395-411.

Noelting, G. (1980a). The development of proportional reasoning and the ratio concept. Part 1--Differentiation of stages. Educational Studies in Mathematics, 11, 217-253.

Noelting, G. (1980b). The development of proportional reasoning and the ratio concept. Part 2--Problem-structure at successive stages; Problem-solving strategies and the mechanism of adaptive restructuring. Educational Studies in Mathematics, 11, 331-363.

Post, T. R., Behr, M. J., Lesh, R., & Wachsmuth, I. (1985). Selected results from the rational number project. In L. Streefland (Ed.), Proceedings of the Ninth International Conference for the Psychology of Mathematics Education. Vol. 1: Individual contributions (pp. 342-351). Noordwijkerhout, The Netherlands.

Post, T. R., Harel, G., Behr, M., & Lesh, R. (1988). Intermediate teachers' knowledge of rational number concepts. In E. Fennema, T. P. Carpenter, & S. J. Lamon (Eds.), Integrating Research on Teaching and Learning Mathematics (pp. 194-217). Madison: Wisconsin Center for Education Research.

Post, T. R., Wachsmuth, I., Lesh, R., & Behr, M. (1985). Order and equivalence of rational numbers: A cognitive analysis. Journal for Research in Mathematics Education, 16, 18-36.

Pothier, Y. (1981). Partitioning: Construction of rational numbers in young children. Unpublished doctoral thesis. University of Alberta, Edmonton.

Rupley, W. H. (1981). The effects of numerical characteristics on the difficulty of proportional problems. Unpublished doctoral dissertation, University of California, Berkeley.

Schwartz, J. (1988). Intensive quantity and referent transforming arithmetic operations. In J. Hiebert & M. Behr, Eds., Number concepts and operations in the middle grades (pp. 41-52). Reston, VA: National Council of Teachers of Mathematics.

Steffe, L. P. (1969). Proportionality. Unpublished manuscript. University of Georgia, Athens.

Steffe, L. P. (1990). Children's multiplying and dividing schemes: An overview. Manuscript submitted for publication.

Steffe, L. P. & Cobb, P. (1988). Construction of arithmetical meanings and strategies. New York: Springer-Verlag.

Steffe, L. P., & Parr, R. B. (1968). The development of the concepts of ratio and fraction in the fourth, fifth, and sixth years of the elementary school. Technical Report No. 49. Madison, WI: Wisconsin Research and Development Center for Cognitive Learning.

Streefland, L. (1983). The long term learning process for ratio. In R. Hershkowitz (Ed.), Proceedings of the Seventh Conference for the Psychology of Mathematics Education (pp. 182-187). Rehovot, Israel.

Streefland, L. (1984). Search for the roots of ratio: Some thoughts on the long term learning process (Towards...a theory). Part I: Reflections on a teaching experiment. Educational Studies in Mathematics, 15, 327-348.

Streefland, L. (1985). Search for the roots of ratio: Some thoughts on the long term learning process (Towards...a theory). Part II: The outline of the long term learning process. Educational Studies in Mathematics, 16, 75-94.

Tourniaire, F. (1986). Proportions in elementary school. Educational Studies in Mathematics, 17, 401-412.

Treffers, A., & Goffree, F. (1985). Rational analysis of realistic mathematics education--the Wiskobas Program. In L. Streefland (Ed.), Proceedings of the Ninth International Conference for the Psychology of Mathematics Education (pp. 97-121). Noordwijkerhout, The Netherlands.

Van den Brink, J., & Streefland, L. (1979). Young children (6-8)-Ratio and proportion. Educational Studies in Mathematics, 10, 403-420.

| | + − | × ÷ | ratio |
|---|---|---|---|
| Modeling and Counting | counting objects<br>1,2,3,4,5<br><br>subitizing: the visual formation of composite units<br> 5 | segmented counting: forming experential composites<br>How many 3's in 12?<br>1,2,3; 4,5,6; 7,8,9; 10,11,12 | counting triggered by pictorial presentation; forming experential ratios: matching 3 aliens and 5 food pellets |
| Composing | constructing a numerical composite<br> <br>5 6 7 8 | early multiplication scheme: repetitively forming and counting 3-units<br>1-2-3 1<br>1-2-3 2<br>1-2-3 3<br>1-2-3 4<br>4 × 3 | repetitively matching and counting 3-units and 5-units<br>1-2-3 1-2-3-4-5 1<br>1-2-3 1-2-3-4-5 2<br>1-2-3 1-2-3-4-5 3 |
| Abstracting | constructing an abstract numerical composite: thinking of one 5-unit without considering its constituent one-units | forming iterable units: 3 is stripped of its composite quality<br>3 is 1<br>6 is 2<br>9 is 3<br>12 is 4<br>double counting | Ratios were not iterable for the children interviewed. 3 ⊖ 5 was not a single entity. |
| Relating | PPW: the whole is a numerical composite whose parts are also composites | PPW: the distributive property: eight 4-units can be achieved by adding five 4-units and three 4-units; parts are multiples of the same composite units | PPW: a multiple of the whole is composed of multiples of different composite units.<br>5(8-units) = 5(3-units) + 5(4-units) + 5(1-units) |

Figure 1. The increasing complexity of the unitizing process as revealed by children's thinking in the additive and multiplicative conceptual fields.

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